

Motion of systems with Variable Mass (Rocket Motion)

- The motion of a simple rocket is an interesting application of elementary Newtonian dynamics, however we want to include more complicated rocket with exhaust mass and multiple rocket stages.

- The two cases we examine are rocket motion in free space

$$F_{\text{ext}} = 0$$

- The vertical ascent of rocket ~~motion~~ under gravity.

Rocket Motion In free Space

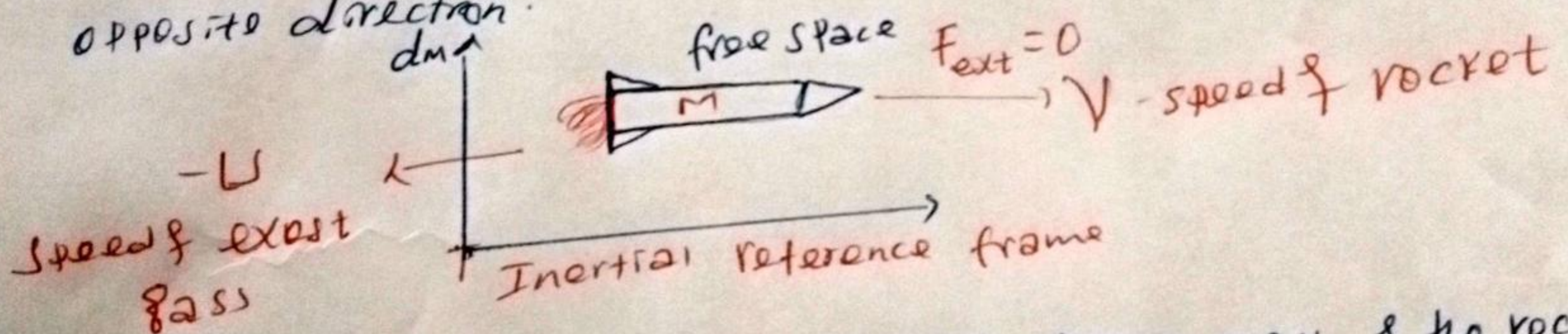
by Merkebo. 6

- We assume here that the rocket moves under the influence of no external force.

- The motion of the spaceship (rocket) must depend entirely on its own energy.

i.e. It moves by the reaction of ejecting mass at high velocity.

- To conserve linear momentum, the rocket will have to move in opposite direction.



At some time t the instantaneous total mass of the rocket is m and the instantaneous speed of the rocket is V with respect to an inertial reference frame.

- We are assuming that all the motion is in x -direction and eliminate the vector notation.

During a time interval dt , a positive mass dm is ejected from the engine with the speed U with respect to the rocket.

⊗ Immediately after the mass dm' is ejected, the rocket
the mass and the speed are $m - dm'$ & $V + dV$ respectively.
~ So Initial momentum = MV (at time t)

$$\text{final momentum} = (m - dm')(V + dV) + dm'(V - U)$$

* According to Conservation of Linear momentum

$$P_{in} = P_{final}$$

$$P(t) = P(t+dt)$$

$$mV = (m - dm')(V + dV) + dm'(V - U)$$

momentum rocket

momentum of
exhaust gas

$$\cancel{mV} = \cancel{mV} + m dV - \cancel{dm' \cdot V} - \cancel{dm' dV} + \cancel{dm' V} - dm' U$$

$$0 = m dV - dm' U$$

$$\cancel{dV} = \cancel{U} \frac{dm'}{m}$$

~ we have consider dm' to be positive mass ejected
from the rocket. where as the change in mass of the rocket
it self is dm .

$$dm = -dm'$$

substituting

$$dV = -U \frac{dm}{m}$$

Integrating the result will be

$$\int_{V_0}^V dV = -U \int_{m_0}^m \frac{dm}{m}$$

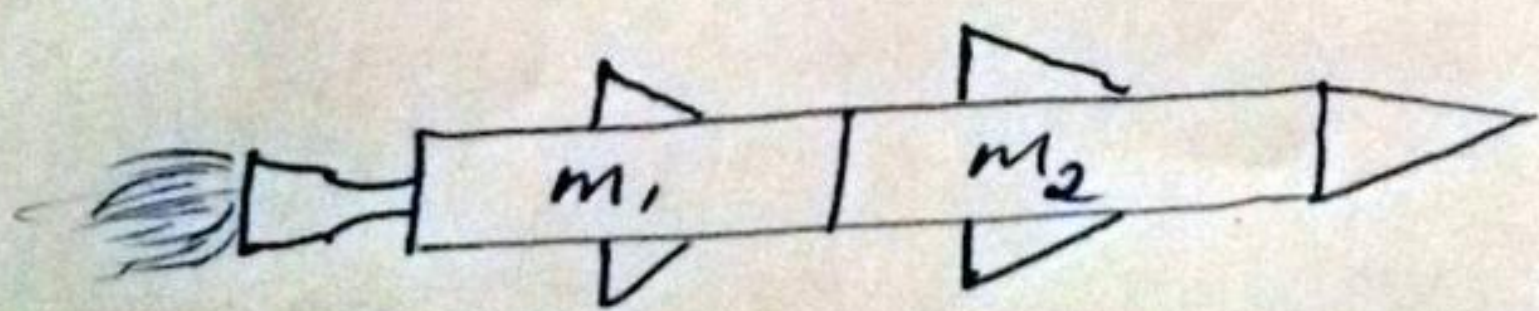
$$\int_{v_0}^v \frac{1}{v} dv = -U \int_{m_0}^m \frac{1}{m} dm = -U \ln \left(\frac{m}{m_0} \right)$$

$$v - v_0 = -U \ln \left(\frac{m}{m_0} \right)$$

$$v = v_0 - U \ln \left(\frac{m}{m_0} \right)$$

$$v = U_0 + U \ln \left(\frac{m_0}{m} \right)$$

- The exhaust Velocity U is Constant. Thus, to maximize the rocket speed, we need to maximize the exhaust Velocity U and the ratio m_0/m .
- The ratio m_0/m limits the terminal speed so the rocket must be multistage.
- If the fuel container itself is (ejected out) after its fuel has been burned. the mass of the remaining rockets is become less.
- The rocket can contain two or more fuel containers each of which can be jettisoned.



for example let's say

m_0 = Initial total mass of rocket

$m_1 = m_a + m_b$

m_a = mass of the first stage payload (fuel)

m_b = mass of first stage fuel containers.

V_1 = Terminal speed of the first stage of burnout after all fuel is burned.

$$V_1 = U_0 + V \left(\ln \frac{m_0}{m_1} \right)$$

* At burnout, the terminal speed V_1 of the first stage is reached, and the mass m_b is released into space, next, the second-stage rocket is ignited with the same exhaust velocity and we have



m_0 = Initial total mass of rocket second stage

$$m_2 = m_c + m_d$$

m_c = mass of the second-stage payload (fuel)

m_d = mass of the second-stage fuel container

U_1 = Initial speed of second stage.

V_2 = The terminal speed of second stage at burnout.

$$V_2 = U_1 + U_{12} \left(\frac{m_2}{m_2} \right)$$

$$V_2 = V_0 + U \ln \left(\frac{m_0}{m_1} \right) + U \ln \left(\frac{m_0}{m_2} \right)$$

$$V_2 = V_0 + U \ln \left(\frac{m_0 m_2}{m_1 m_2} \right)$$

- The product $\left(\frac{m_0 m_2}{m_1 m_2} \right)$ can be made larger than just $\left(\frac{m_0}{m_1} \right)$.

Vertical Ascent Under Gravity

- The actual motion of rocket attempting to leave earth's gravitational field is quite complicated.
- we begin by making several assumption
 - the rocket will have only vertical motion
 - no horizontal components
 - neglecting air resistance

* Assume that the acceleration of gravity is constant with height.

* Assume that the burn rate of the fuel is constant, we can use the results of the previous case of rocket motion in free space, but no longer have

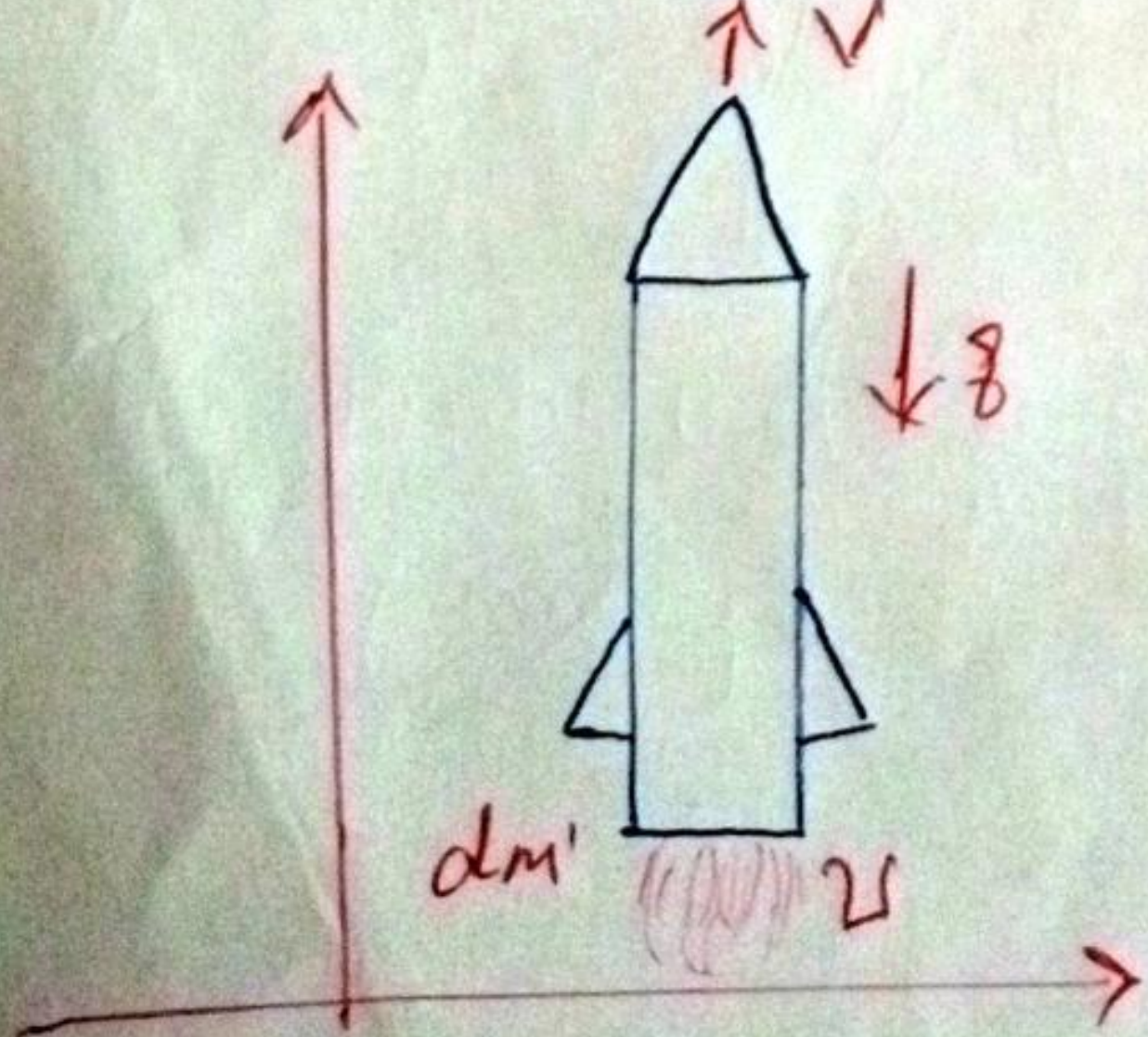
$$F_{ext} = 0$$

we again have dm' as positive, with

$$dm = -dm'$$

The external force can be express as

$$F_{ext} = \frac{d(mv)}{dt}$$



$$F_{ext} dt = d(mv) = dP = P(t+dv) - P(t)$$

Very small change of momentum

- For the rocket system, we found the initial and final momentum for rocket motion in free space.

$$p(t+dt) - p(t) = m dv + v dm$$

In free space $F_{ext} = 0$ but in vertical ascent

$$F_{ext} = -mg \quad \dots \dots \dots \rightarrow (**)$$

- Combining the two equations

$$\text{i.e. } F_{ext} dt = -mg \frac{dt}{dt} = m \frac{dv}{dt} + v \frac{dm}{dt}$$

$$-mg = m \ddot{v} + v \dot{m}$$

- we assume the fuel burn rate is constant, let

$$\dot{m} = \frac{dm}{dt} = -\alpha, \quad \alpha > 0$$

$$\text{so } -mg = m\ddot{v} - \alpha v$$

$$v = \alpha v - mg$$

$$\cancel{dt} \frac{dv}{\cancel{dt}} = (\alpha v - mg) dt$$

$$dv = (\alpha v - mg) dt = \left(\frac{\alpha v}{\alpha m} - g \right) dt$$

- To eliminate time

$$dt = dm / \alpha$$

$$dv = \left(\frac{\alpha v}{m} - g \right) \left(-\frac{dm}{\alpha} \right)$$

$$dv = \left(\frac{g}{\alpha} - \frac{v}{m} \right) dm$$

Assume the initial and final values of the Velocity to be 0 and U respectively and the mass m_0 and m respectively. So that

$$\int_0^U dv = \int_{m_0}^m \left(\frac{g}{2} - \frac{U}{m} \right) dm$$

$$V \int_0^U = \int_{m_0}^m \frac{g}{2} dm - \int_{m_0}^m \frac{U}{m} dm$$

$$U - 0 = \frac{g}{2} (m - m_0) - U \ln \left(\frac{m}{m_0} \right)$$

- To find the time: (At burnout)

$$dm = -2 dt$$

$$dm \int_{m_0}^m = \int_0^t 2 dt$$

$$m - m_0 = -2t$$

$$m_0 - m = 2t$$

$$U = -\frac{g}{2} (2t) = U = -gt + U \ln \left(\frac{m_0}{m} \right)$$

- The height of the rocket at burn out

$$U = \frac{dv}{dt}$$

$$dt \frac{dy}{dt} = (-gt + U \ln \left(\frac{m_0}{m} \right)) dt$$

$$\int dy = \int \left[-gt + U \ln \left(\frac{m_0}{m} \right) \right] dt$$

$$y + c = - \int gt dt + \int U \ln \left(\frac{m_0}{m} \right) dt$$

$$, dt = - \frac{dm}{2}$$

$$Y+C = - \int g t dt + \int U \ln \left(\frac{m_0}{m} \right) \frac{dm}{2}$$

$$Y+C = - \int g t dt - \frac{U}{2} \int \ln \left(\frac{m_0}{m} \right) dm$$

$$Y+C = -\frac{1}{2} g t^2 - \frac{U}{2} \int \ln \left(\frac{m_0}{m} \right) dm$$

Since $\int \ln \frac{a}{x} dx = x + x \ln \left(\frac{a}{x} \right)$

$$Y+C = -\frac{1}{2} g t^2 \left[m + m \ln \left(\frac{m_0}{m} \right) \right]$$

* To evaluate "C" using $Y=0$ when $t=0, m=m_0$

$$0+C = -\frac{U}{2} \left(m_0 + m \ln \left(\frac{m_0}{m} \right) \right)$$

$$C = -\frac{U}{2} \left(m_0 + m \ln(1) \right), \quad \ln(1) = 0$$

$$C = -\frac{U}{2} m_0$$

$$Y + \left(-\frac{U}{2} m_0 \right) = -\frac{1}{2} g t^2 - \frac{U m}{2} - \frac{U m}{2} \ln \left(\frac{m_0}{m} \right)$$

$$Y = -\frac{1}{2} g t^2 + \frac{U m_0}{2} - \frac{U m}{2} - \frac{U m}{2} \ln \left(\frac{m_0}{m} \right)$$

$$Y = -\frac{1}{2} g t^2 + \frac{U}{2} (m_0 - m) - \frac{U m}{2} \ln \left(\frac{m_0}{m} \right)$$

$$Y = Y_B, \quad t = t_B, \quad 2 t_B = (m_0 - m)$$

$$Y_B = U t_B - \frac{1}{2} g t_B^2 - \frac{m v}{2} \ln \left(\frac{m_0 - m}{m} \right)$$

light of the rocket
burnout time.

11.2-8

Collision of Particle

- for an isolated system of particles with no external force acting on it, the total linear momentum is conserved.
- If the mass of the particle are fixed the velocity of the center of mass is constant
- The reference frame that moves with the center of mass is therefore inertial and so we can use newtonian mechanics in the reference system.

we use the following notation:

$$\left. \begin{matrix} m_1 = \\ m_2 = \end{matrix} \right\} \text{mass of } \left\{ \begin{matrix} \text{moving} \\ \text{stuck} \end{matrix} \right\} \text{particle.}$$

In general, primed quantities refer to the CM (center of mass) system:

$$\left. \begin{matrix} U_1 = \text{initial} \\ V_1 = \text{final} \end{matrix} \right\} \text{Velocity of } m_1 \text{ in the LAB system}$$

$$\left. \begin{matrix} U_1' = \text{initial} \\ V_1' = \text{final} \end{matrix} \right\} \text{Velocity of } m_1 \text{ in the CM system}$$

and similarly for U_2, V_2, U_2' and V_2' (but $U_2 = 0$):

$$T_0 =$$

$$T_0 = \text{Total initial kinetic energy in } \left(\begin{matrix} \text{LAB} \\ \text{CM} \end{matrix} \right) \text{ system}$$

$$\left. \begin{matrix} T_1 \\ T_1' \end{matrix} \right\} = \text{Final kinetic energy of } (m_1) \text{ in } \left\{ \begin{matrix} \text{LAB} \\ \text{CM} \end{matrix} \right\} \text{ system}$$

And similar for T_2 and T_2'

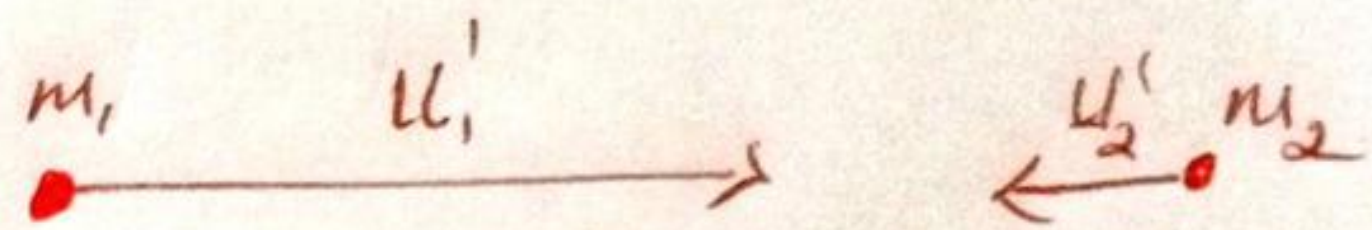
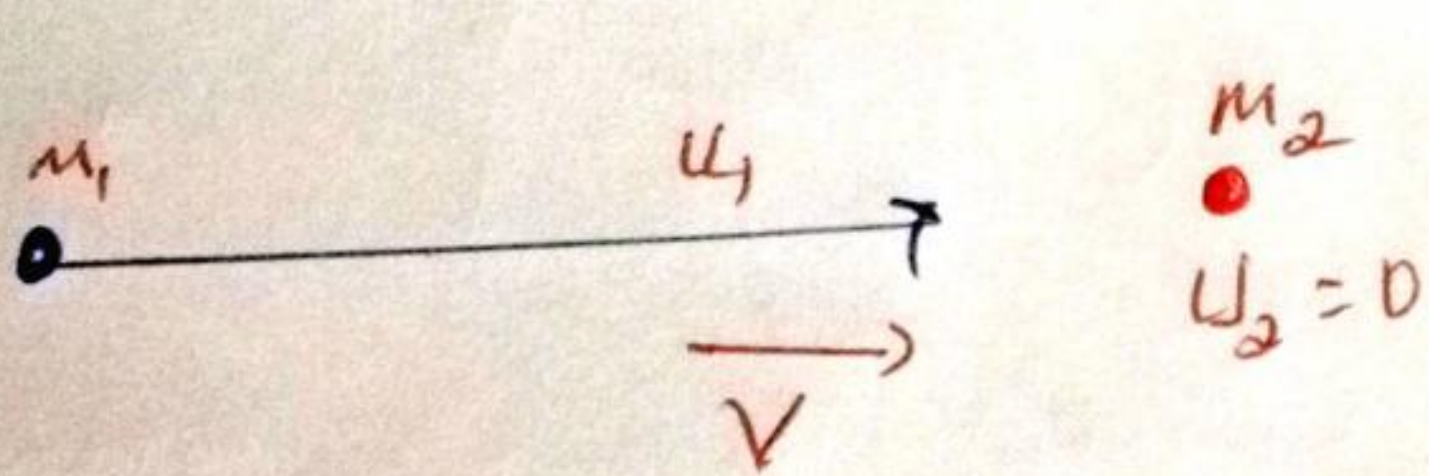
V = Velocity of center of mass in the LAB system

ψ = angle through which m_1 is deflected in the LAB system

ψ' = angle through which m_2 is deflected in the LAB system

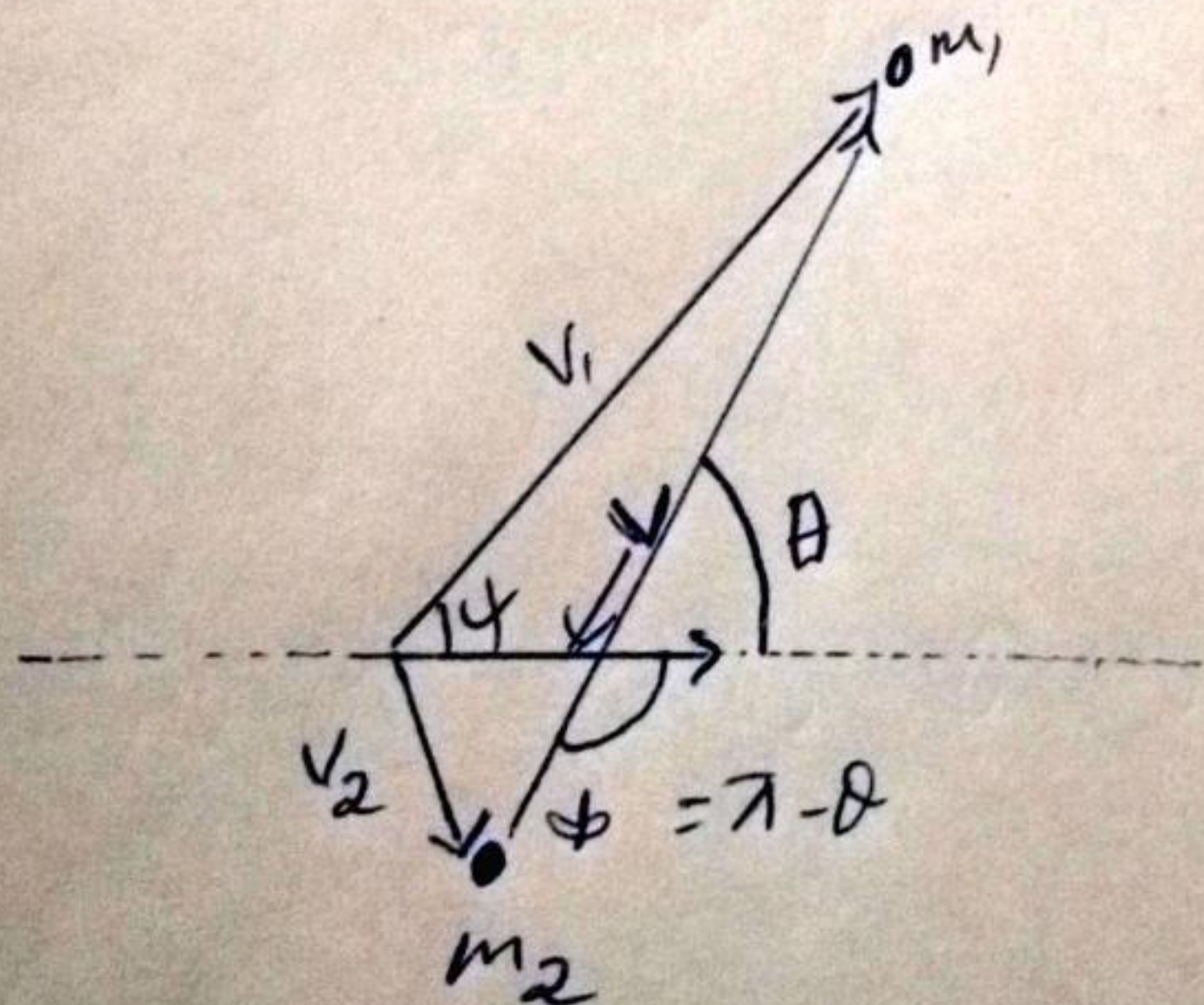
θ = angle through which m_1 and m_2 are deflected in the CM system.

Let us illustrate the geometry of elastic collision in both LAB and CM system.

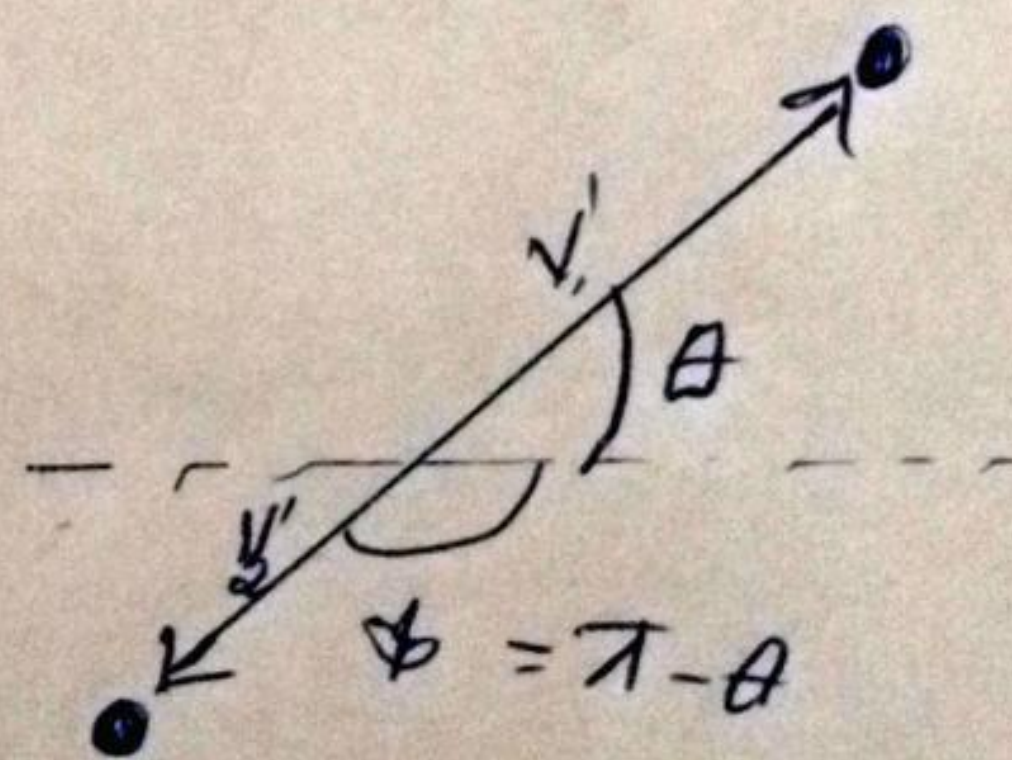


b, final condition

a, Initial condition



c, Final condition



d, Final condition

Let us consider the problem quantitatively
the center of mass

$$m_1 \vec{r}_1 + m_2 \vec{r}_2 = M \vec{R}$$

Differentiating with respect to the time, we find

$$m_1 \vec{u}_1 + m_2 \vec{u}_2 = M \vec{V}$$

But $u_2 = 0$ and $M = m_1 + m_2$; the center of mass must
therefore be moving (in the LAB system) towards m_2 with a velocity

$$V = \frac{m_1 u_1}{m_1 + m_2}$$

By the same reasoning, but m_2 is initially at rest, the
CM speed of m_2 must just equal to V :

$$u_2' = V = \frac{m_1 u_1}{m_1 + m_2}$$

Note:- however, that the motion and the velocity are
opposite in direction and that vectorially $u_2' = -V$.

Elastic Collision

H.M
Do the problems in
your text book!!!